SHENTON COLLEGE

# ATMAS Mathematics Specialist

## 2018 Test 2 Calculator Free

Name: SOLUTIONS

Time Allowed : 50 minutes

| Marks | /52 |
|---|---|

***Materials allowed:*** No special materials.

***All necessary working and reasoning must be shown for full marks.***
*Where appropriate, answers should be given in exact values.*
*Marks may not be awarded for untidy or poorly arranged work.*

**1** If $f(x) = \frac{1}{x-1}$ and $g(x) = x^2 - 3$,

Determine the domain and range of the composition $f(g(x))$. (5)

$g(x)$ D: $x \in \mathbb{R}$ $x \neq 2, x \neq -2$

R: $y \geq -3$ $y \neq 1$

$f(x)$ D: $x \geq -3$ and $x \neq 1$

R: $y > 0,\ y \leq -\frac{1}{4}$

✓ natural D&R $g(x)$
✓ R $g(x) \to$ D $f(x)$
✓ natural D $f(x)$
✓ backtrack restrictions on $g(x)$
✓ final D&R.

Range of $f(x)$

$x \to -1^+,\ y \to \infty$

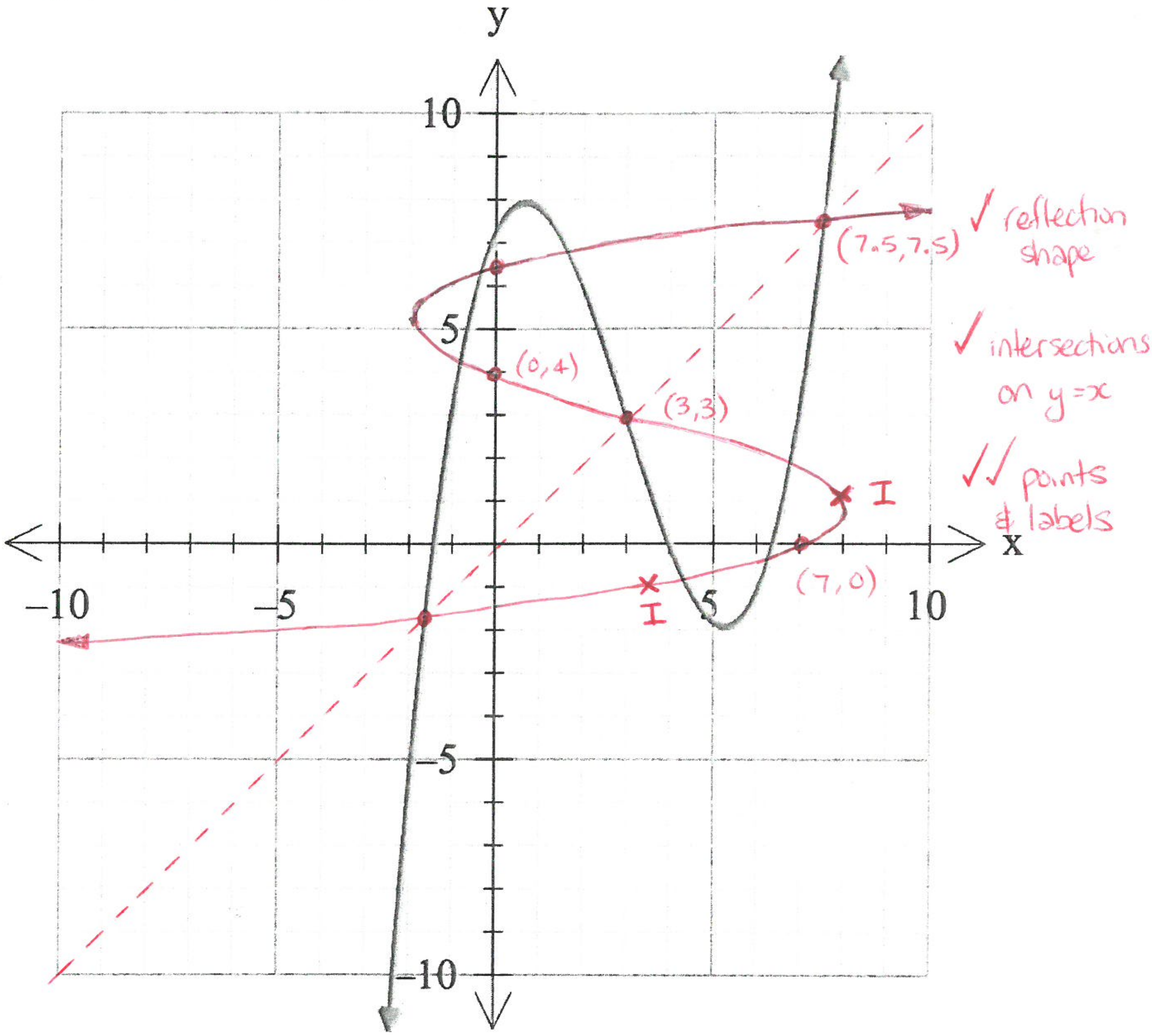
$x \to -1^-,\ y \to -\infty$

$x \to \infty,\ y \to 0$

$x = -3,\ y = \frac{-1}{4}$

$f(g(x))$ $\begin{cases} D: x \neq 2, x \neq -2 \\ R: y > 0,\ y \leq -\frac{1}{4} \end{cases}$

**2** The graph below shows $y = f(x)$.




a) Add a sketch of $f^{-1}(x)$ to the axes above, indicating at least 3 key points. (4)

b) Explain why $f^{-1}(x)$ is not a function. (1)

One-to-many,

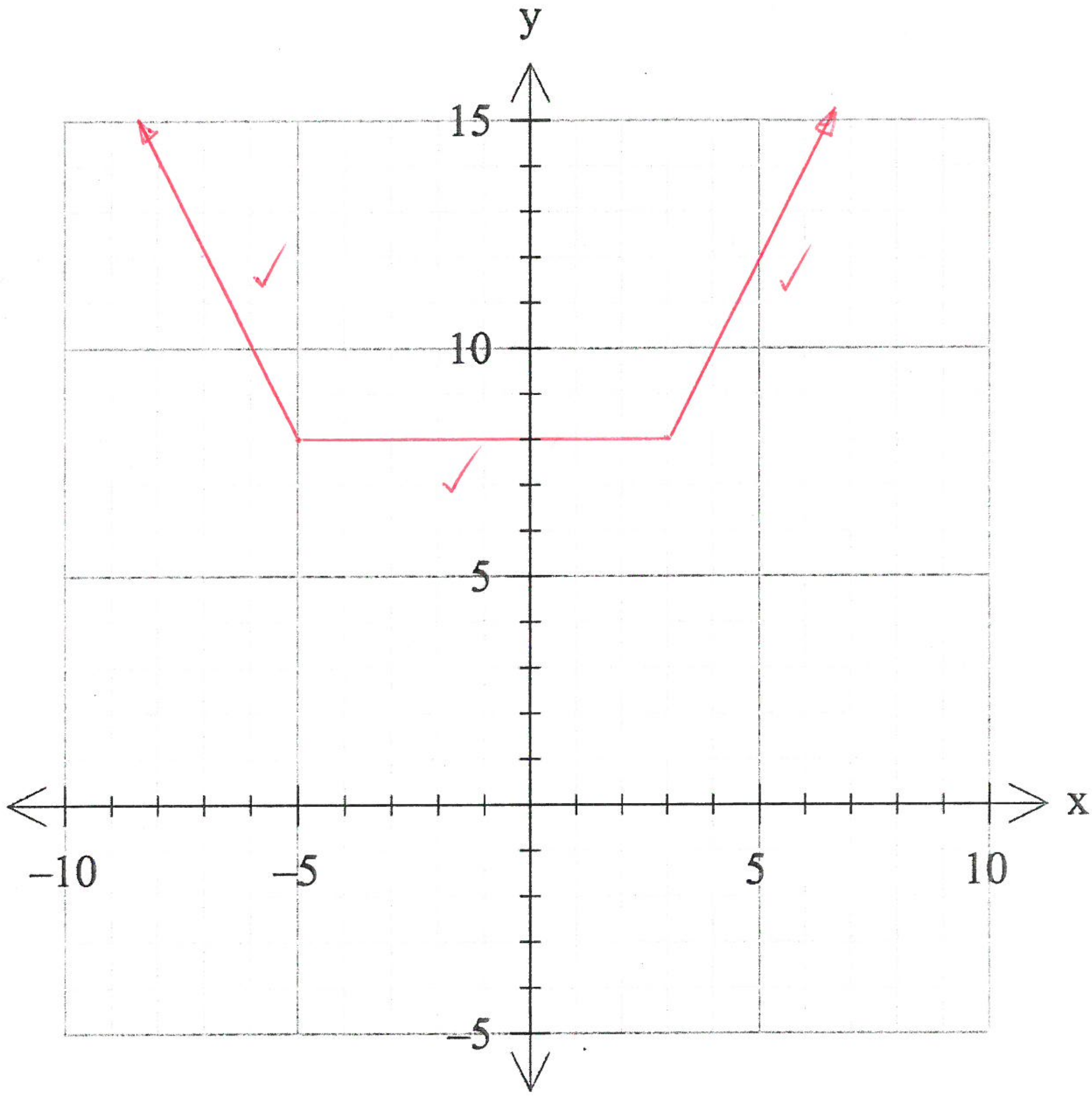
$x$ values between $-2 \leq x \leq 8$ have non-unique $y$ values

c) Mark on you sketch of $f^{-1}(x)$ the points where it would intersect with $\frac{1}{f^{-1}(x)}$. (2)

(Do not graph $\frac{1}{f^{-1}(x)}$.)

Marked as $\overset{I}{\times}$ on graph. $|y| = 1$

**3** a) Sketch the graph of $y = |x - 3| + |x + 5|$ (4)




$x < -5$

$-(x-3) - (x+5)$

$= -2x - 2$

$-5 \le x \le 3$

$(x-3) + (x+5)$

$= 8$

$x > 3$

$x - 3 + x + 5$

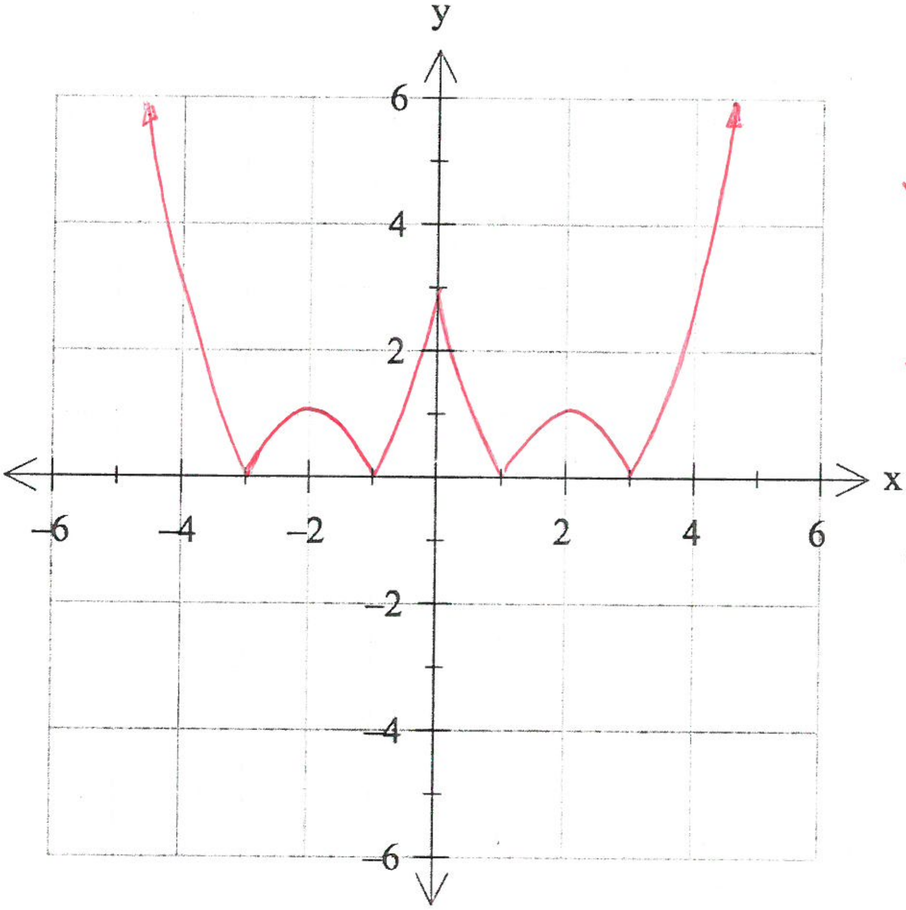
$= 2x + 2$

✓ Some working, various forms possible

b) Hence or otherwise solve $|x - 3| + |x + 5| = 12$ (2)

$x = -7$ or $x = 5$.

**4** If $f(x) = (x-2)^2 - 1$, sketch $|f(|x|)|$ on the axes below. (3)




✓ reflection $|f(x)|$

✓ reflection $f(|x|)$

✓ location & shape

**5** a) Determine a vector equation for the line parallel to $5\boldsymbol{i} - 4\boldsymbol{k}$ and passing through the point $-3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$. (2)

$$\underset{\sim}{r} = (-3+5\lambda)\underset{\sim}{i} + 2\underset{\sim}{j} + (1-4\lambda)\underset{\sim}{k}$$

✓ line

✓ notation.

b) Show whether the line from part a) intersects with the line $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$ (3)

By component

$\underset{\sim}{i}$ $-3+5\lambda = -1-2\mu$

$\underset{\sim}{j}$ $2 = 1+\mu$

$\underset{\sim}{k}$ $1-4\lambda = 3-3\mu$

$\underset{\sim}{j} \Rightarrow \mu = 1$

$\underset{\sim}{i} \Rightarrow \lambda = 0$

$\underset{\sim}{k}$ $1-4(0) \neq 3-3(1)$

$\Rightarrow$ no intersection.

✓ component equations

✓ solve 2

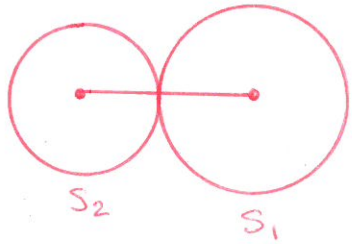
✓ check 3rd

**6** Two spheres are defined by the equations $S_1: \left|\boldsymbol{r} - \begin{pmatrix}-3\\5\\-4\end{pmatrix}\right| = 4$ and $S_2: \left|\boldsymbol{r} - \begin{pmatrix}-1\\-1\\-7\end{pmatrix}\right| = 3$ (3)

Determine whether or not the spheres touch, and if they do, describe the nature of their contact.

$S_1$ centre $\begin{pmatrix}-3\\5\\-4\end{pmatrix}$

$S_2$ centre $\begin{pmatrix}-1\\-1\\-7\end{pmatrix}$




✓ magnitude of centre - centre

✓ radius sum

✓ interpret

Vector from $S_2 \to S_1$ centres

$r_{S_1} + r_{S_2} = 7$

$$\begin{pmatrix}-3\\5\\4\end{pmatrix} - \begin{pmatrix}-1\\-1\\-7\end{pmatrix} = \begin{pmatrix}-2\\6\\3\end{pmatrix}$$

$$\left|\begin{pmatrix}-2\\6\\3\end{pmatrix}\right| = 7$$

$\Rightarrow$ The spheres touch at only one point

**7** A plane contains the points given by the position vectors $\begin{pmatrix}1\\6\\-1\end{pmatrix}$, $\begin{pmatrix}5\\8\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\2\\2\end{pmatrix}$.

a) Write a vector equation for the plane. (3)

$$\begin{pmatrix}5\\8\\-1\end{pmatrix} - \begin{pmatrix}1\\6\\-1\end{pmatrix} = \begin{pmatrix}4\\2\\0\end{pmatrix}$$

✓ direction vector.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} - \begin{pmatrix}1\\6\\-1\end{pmatrix} = \begin{pmatrix}1\\-4\\3\end{pmatrix}$$

✓ second direction vector with different parameter

$$\begin{pmatrix}5\\8\\-1\end{pmatrix} - \begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}3\\6\\-3\end{pmatrix}$$

✓ use of

$$\underset{\sim}{p} = \begin{pmatrix}1\\6\\-1\end{pmatrix} + \lambda\begin{pmatrix}4\\2\\0\end{pmatrix} + \mu\begin{pmatrix}1\\-4\\3\end{pmatrix}$$

b) Write the Cartesian equation for the plane. (4)

$$\begin{pmatrix}4\\2\\0\end{pmatrix} \times \begin{pmatrix}1\\-4\\3\end{pmatrix}$$

$$\begin{array}{cccccc} i & j & k & i & j & k \\ 4 & 2 & 0 & 4 & 2 & 0 \\ 1 & -4 & 3 & 1 & -4 & 3 \end{array}$$

$$6i - 0i + 0j - 12j - 16k - 2k$$

$$= \begin{pmatrix}6\\-12\\-18\end{pmatrix}$$

$$6\begin{pmatrix}1\\-2\\-3\end{pmatrix} \cdot \begin{pmatrix}2\\2\\2\end{pmatrix}$$

$$= 6(2 - 4 - 6)$$

$$= -48$$

✓ identifies cross product

✓ normal

✓ dot product for constant

✓ Cartesian form.

$$6x - 12y - 18z + 48 = 0$$

or $x - 2y - 3z + 8 = 0$

c) Give the equation of a line parallel to the plane and passing through the point $\begin{pmatrix}-3\\1\\5\end{pmatrix}$. (1)

Any direction from part (a) and using $\begin{pmatrix}-3\\1\\5\end{pmatrix}$.

**8** Sketch the following rational functions.

a) $y = \dfrac{2x^2 - 7x + 4}{2x - 1}$ (4)

$$y = \frac{x(2x-1) - 3(2x-1) + 1}{2x-1}$$

$$= x - 3 + \frac{1}{2x-1}$$

$$\frac{dy}{dx} = 1 - \frac{2}{(2x-1)^2}$$

$\frac{dy}{dx} = 0$ (stationary points)

$$\Rightarrow 1 = \frac{2}{(2x-1)^2}$$

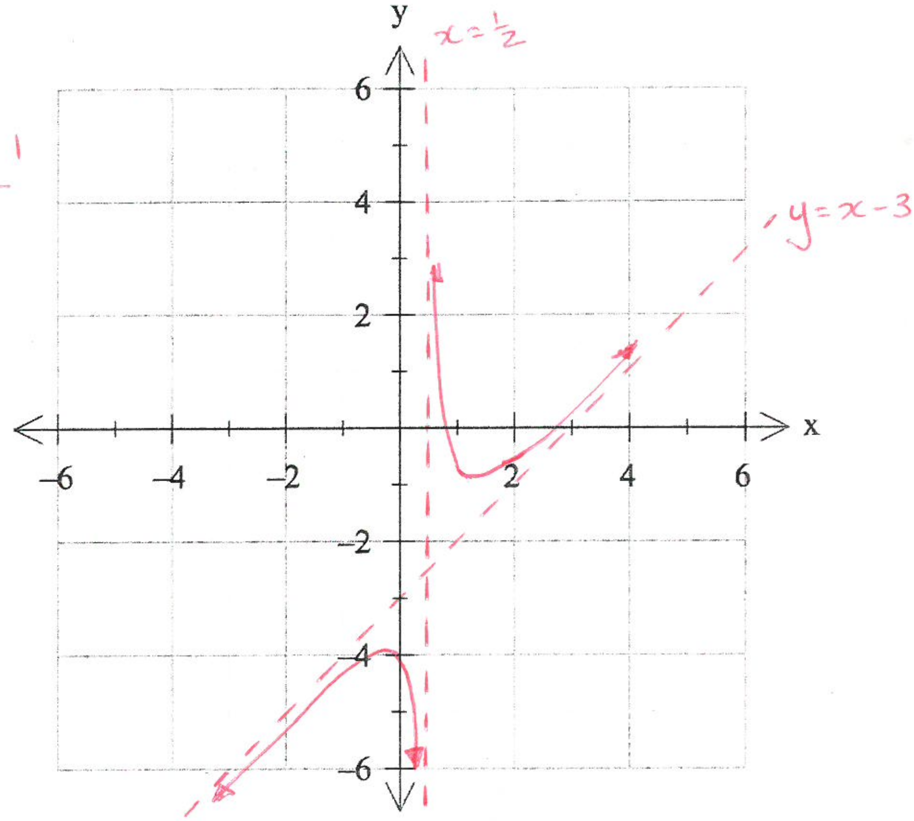
$$2x - 1 = \pm\sqrt{2}$$

$$x = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$$

$x \to \infty,\ y \to \infty$

$x \to -\infty,\ y \to \frac{\infty^2}{-\infty}$

$\to -\infty$




✓ vertical asymptote.

✓ horizontal asymptote

✓ local extrema found

✓ behaviour at $\pm\infty$

b) $y = \dfrac{9}{x^2 - 2x - 8}$, given that $f''(x) = -\dfrac{54(x^2 - 2x + 4)}{(x^2 - 2x - 8)^3}$ and $f''(1) = -\dfrac{2}{9}$ (5)

$y = \dfrac{9}{(x-4)(x+2)}$

$\dfrac{dy}{dx} = \dfrac{-9(2x-2)}{(x^2-2x-8)^2}$

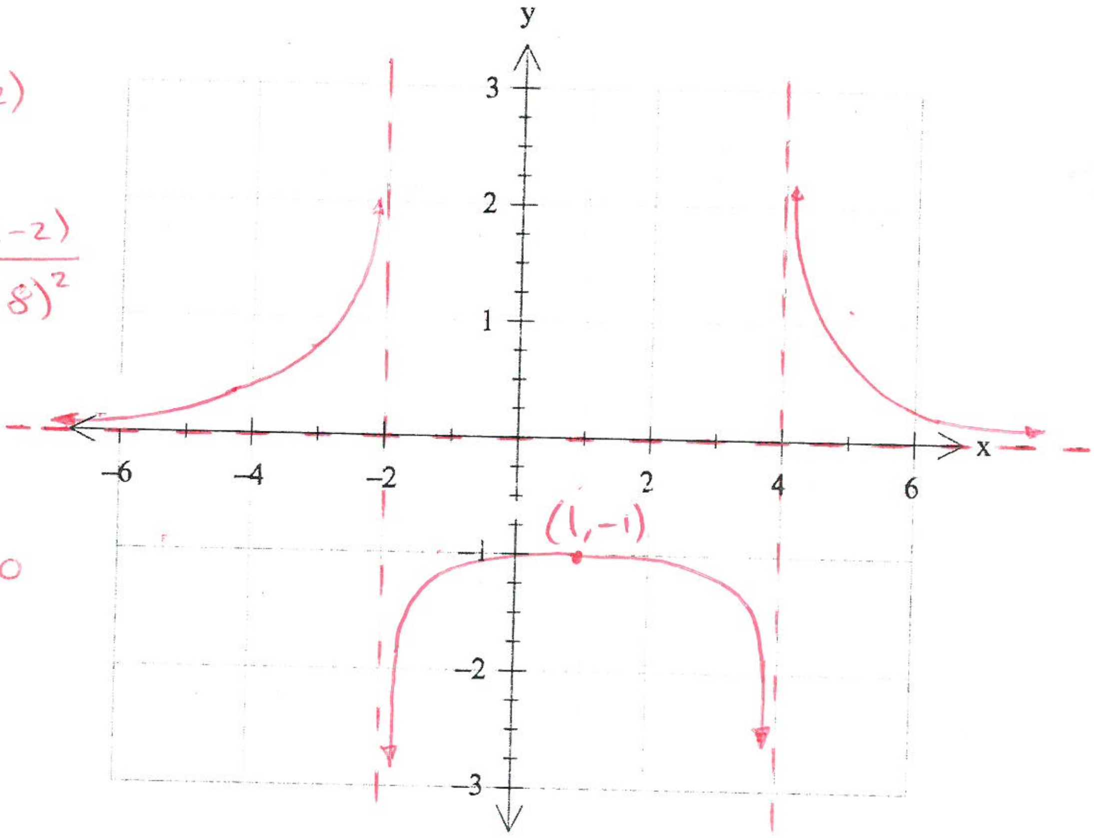
$\dfrac{dy}{dx} = 0$

$\Rightarrow -9(2x-2) = 0$

$x = 1$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=1} < 0$

$\Rightarrow$ local maximum at $(1,-1)$




✓ vertical asymptotes

✓ turning point

✓ behavior of

✓ behavior at

✓ behavior of

$\dfrac{d^2y}{dx^2} = 0$

$\Rightarrow (x^2 - 2x + 4) = 0$

No real solutions

$\therefore$ no inflection points.

$x \to \infty,\ y \to 0^+$

$x \to -\infty,\ y \to 0^+$

$x \to -2^-,\ y \to \infty$

$x \to 4^+,\ y \to \infty$

**9** Two particles are moving through free space. Particle A starts at position $\begin{pmatrix}-3\\1\\7\end{pmatrix}$ and is moving with constant velocity $\begin{pmatrix}1\\0\\2\end{pmatrix}$. Particle B is initially at $\begin{pmatrix}5\\-6\\-8\end{pmatrix}$ and moving with velocity $\begin{pmatrix}-3\\1\\3\end{pmatrix}$. All distances are in kilometres and time is in seconds. Determine the time at which the two particles are closest to each other, and the size of that minimum separation. (6)

$$\tilde{r}_A(t) = \begin{pmatrix}-3\\1\\7\end{pmatrix} + \begin{pmatrix}1\\0\\2\end{pmatrix}t$$

$$\tilde{r}_B(t) = \begin{pmatrix}5\\-6\\-8\end{pmatrix} + \begin{pmatrix}-3\\1\\3\end{pmatrix}t$$

$${}_A\tilde{v}_B = \begin{pmatrix}4\\-1\\-1\end{pmatrix}$$

$${}_A\tilde{r}_B(t) = \begin{pmatrix}-8\\7\\15\end{pmatrix} + \begin{pmatrix}4\\-1\\-1\end{pmatrix}t$$

Closest approach when ${}_A\tilde{v}_B \cdot {}_A\tilde{r}_B(t) = 0$

$$\begin{pmatrix}-8+4t\\7-t\\15-t\end{pmatrix} \cdot \begin{pmatrix}4\\-1\\-1\end{pmatrix} = 0$$

$$-32+16t-7+t-15+t=0$$

$$t=3$$

$${}_A\tilde{r}_B(3) = \begin{pmatrix}-8+12\\7-3\\15-3\end{pmatrix} = \begin{pmatrix}4\\4\\12\end{pmatrix}$$

$$\left|\begin{pmatrix}4\\4\\12\end{pmatrix}\right| = \sqrt{4^2+4^2+12^2} = \sqrt{176} = 4\sqrt{11}$$

Minimum separation of $4\sqrt{11}$ km occurs at $t=3$ seconds.

✓ ${}_A\tilde{v}_B$
✓ ${}_A\tilde{r}_B(t)$
✓ dot product
✓ $t$
✓ ${}_A\tilde{r}_B(3)$
✓ $|{}_A\tilde{r}_B(3)|$